Technical appendix

Following the methodology developed by Wang, Wei, Zhu (2014), consider a simplified input-output model of two countries (r and s) and two sectors (1 and 2) producing homogenous goods, depicted in Table A1:

Table A1: Input-output model of two countries

	Country	Intermediates used				Final demand		Total output
		R		S		Y^r	Y^s	
Country	Sector	1	2	1	2			
r	1	k_{11}^{rr}	k_{12}^{rr}	k_{11}^{rs}	k_{12}^{rs}	y_1^{rr}	y_1^{rs}	x_1^r
	2	k_{21}^{rr}	k_{22}^{rr}	k_{21}^{rs}	k_{22}^{rs}	y_2^{rr}	y_2^{rs}	x_2^r
s	1	k_{11}^{sr}	k_{12}^{sr}	k_{11}^{ss}	k_{12}^{ss}	y_1^{sr}	y_1^{ss}	x_1^s
	2	k_{21}^{sr}	k_{22}^{sr}	k_{21}^{ss}	k_{22}^{ss}	y_2^{sr}	y_2^{ss}	x_2^s
Value added		va_1^r	va_2^r	va_1^s	va_2^s			
Total output		x_1^r	x_2^r	x_1^s	x_2^s			

The elements of the table above can be defined in the following way:

- k_{ij}^{wt} represents the value of intermediate inputs supplied by country w / sector i to country t / sector j;
- x_i^w is the total value of production of country w / sector i;
- va_i^w is value-added of country w / sector i (that is, the difference between what is produced and what is used as intermediates from other country/sectors); and
- y_i^{wt} is the value of final goods produced in country w / sector i and consumed in country t (that is, the difference between what is produced and what is supplied as intermediates to other country/sectors).

Let $a_{ij}^{wt} = k_{ij}^{wt}/x_i^t$ (that is, the share of intermediates supplied by country w / sector i to country t / sector j, over total value of production of country w / sector i). Then, markets clear when total production (supply) of country w / sector i equals total intermediate and final demand, which is expressed as:

$$x_i^w = \sum_{t=\{w,t\}} \sum_{j=\{1,2\}} (a_{ij}^{wt} x_j^t) + \sum_{t=\{w,t\}} y_i^{wt}, \forall w \in \{r,s\}, \forall i \in \{1,2\} \qquad (1)$$

Therefore, in matrix notation, the whole system can be expressed as:

$$\begin{bmatrix} x_1^r \\ x_2^r \\ x_1^s \\ x_2^s \end{bmatrix} = \begin{bmatrix} a_{11}^{rr} & a_{12}^{rr} & a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rr} & a_{22}^{rr} & a_{21}^{rs} & a_{22}^{rs} \\ a_{11}^{sr} & a_{12}^{sr} & a_{11}^{ss} & a_{12}^{ss} \\ a_{21}^{sr} & a_{22}^{sr} & a_{21}^{ss} & a_{22}^{ss} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \\ x_1^s \\ x_2^s \end{bmatrix} + \begin{bmatrix} y_1^{rr} + y_1^{rs} \\ y_2^{rr} + y_2^{rs} \\ y_1^{sr} + y_1^{ss} \\ y_2^{sr} + y_2^{ss} \end{bmatrix} \qquad (2)$$

and in compact form:

$$X = AX + Y \qquad (3)$$

Solving for output as a function of final demand:

$$X = (I - A)^{-1}Y \qquad (4)$$

where I is the identity matrix. A (the "technical coefficient matrix") is invertible if it is of full rank (Miller and Jones, 2009); the so-called Leontief inverse can be alternatively expressed as follows:

$$B = (I - A)^{-1} = A + A^2 + A^3 + \cdots \qquad (5)$$

B is a 4 x 4 matrix; the i^{it} row and the jth column element gives the additional output required from the ith country-sector for the production of an extra unit of final good by the jth country-sector. The interpretation of B as an infinite geometric series of A follows the intuition that when one extra such unit is produced, it requires the use of intermediates as inputs, which themselves also require their own corresponding intermediate input, and so on.

Define a vector V of value-added shares in total production, whose generic element looks like this:

$$v_i^w = \frac{va_i^w}{x_i^w} = 1 - \sum_{j=1}^{2} a_{ji}^{ww} - \sum_{j=1}^{2} a_{ji}^{tw}$$

that is, the share of value-added over total production of country w / sector i equals 1 minus the share of all intermediate inputs used (whether produced domestically or abroad).

Then, rearranging equation (4), total value-added can be obtained as:

$$\hat{V}X = \begin{bmatrix} v_1^r & 0 & 0 & 0 \\ 0 & v_2^r & 0 & 0 \\ 0 & 0 & v_1^s & 0 \\ 0 & 0 & 0 & v_2^s \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \\ x_1^s \\ x_2^s \end{bmatrix} = \begin{bmatrix} v_1^r & 0 & 0 & 0 \\ 0 & v_2^r & 0 & 0 \\ 0 & 0 & v_1^s & 0 \\ 0 & 0 & 0 & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} & b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rr} & b_{22}^{rr} & b_{21}^{rs} & b_{22}^{rs} \\ b_{11}^{sr} & b_{12}^{sr} & b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{sr} & b_{22}^{sr} & b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{rr} + y_1^{rs} \\ y_2^{rr} + y_2^{rs} \\ y_1^{sr} + y_1^{ss} \\ y_2^{sr} + y_2^{ss} \end{bmatrix} = \hat{V}BY \qquad (5)$$

where $\hat{V}$ is vector V diagonalised. The matrix in the right-hand side, thus, decomposes value-added created by a country-industry pair according to which country-industry uses it in the production of its final goods.

In order to obtain-value added according to where it is sold (absorbed), we need to break down the vector of final good demand in the various possible absorption destinations.

Consider, for simplicity, only the first row in (5):

$$v_1^r x_1^r = v_1^r b_{11}^{rr}(y_1^{rr} + y_1^{rs}) + v_1^r b_{12}^{rr}(y_2^{rr} + y_2^{rs}) + v_1^r b_{11}^{rs}(y_1^{sr} + y_1^{ss}) + v_1^r b_{12}^{rs}(y_2^{sr} + y_2^{ss}) \quad (6)$$

x_1^r (total output in country r / sector 1) can be decomposed in:

- x_{11}^{rr} (output in country r / sector 1, used in the production of final goods of sector 1, absorbed in country r);
- x_{12}^{rr} (output in country r / sector 1, used in the production of final goods of sector 2, absorbed in country r);
- x_{11}^{rs} (output in country r / sector 1, used in the production of final goods of sector 1, absorbed in country s); and
- x_{12}^{rs} (output in country r / sector 1, used in the production of final goods of sector 2, absorbed in country s).

Taking this into account, canceling out v_1^r, and rearranging terms of (6), we obtain:

$$\begin{aligned} x_{11}^{rr} + x_{12}^{rr} + x_{11}^{rs} + x_{12}^{rs} \\ &= (b_{11}^{rr} y_1^{rr} + b_{11}^{rs} y_1^{sr}) + (b_{12}^{rr} y_2^{rr} + b_{12}^{rs} y_2^{sr}) \\ &+ (b_{11}^{rr} y_1^{rs} + b_{11}^{rs} y_1^{ss}) + (b_{12}^{rr} y_2^{rs} + b_{12}^{rs} y_2^{ss}) \end{aligned} \tag{7}$$

This means that x_{11}^{rr} (for example) is equal to the sum of:

- country r / sector 1 output, used by <u>country r</u> / sector 1 as an intermediate in the production of final goods sold in r ($b_{11}^{rr} y_1^{rr}$); plus

- country r / sector 1 output, used by <u>country s</u> / sector 1 as an intermediate in the production of final goods sold in r ($b_{11}^{rs} y_1^{sr}$)

The cases of x_{12}^{rr}, x_{11}^{rs}, and x_{12}^{rs} are analogous.

In practice, this decomposition allows us to identify the final destination (in terms of where the final good is sold) of value added created by one country/sector. In particular, we can identify how much value is dependent on external demand (in this simplified case, the other country), and which sectors are dependent the most.

Finally, aggregating all equations in (5) in matrix notation, we obtain:

$$\hat{V}\tilde{X} = \begin{bmatrix} v_1^r & 0 & 0 & 0 \\ 0 & v_2^r & 0 & 0 \\ 0 & 0 & v_1^s & 0 \\ 0 & 0 & 0 & v_2^s \end{bmatrix} \begin{bmatrix} x_{11}^{rr} & x_{12}^{rr} & x_{11}^{rs} & x_{12}^{rs} \\ x_{21}^{rr} & x_{22}^{rr} & x_{21}^{rs} & x_{22}^{rs} \\ x_{11}^{sr} & x_{12}^{sr} & x_{11}^{ss} & x_{12}^{ss} \\ x_{21}^{sr} & x_{22}^{sr} & x_{21}^{ss} & x_{22}^{ss} \end{bmatrix} = \begin{bmatrix} v_1^r & 0 & 0 & 0 \\ 0 & v_2^r & 0 & 0 \\ 0 & 0 & v_1^s & 0 \\ 0 & 0 & 0 & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} & b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rr} & b_{22}^{rr} & b_{21}^{rs} & b_{22}^{rs} \\ b_{11}^{sr} & b_{12}^{sr} & b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{sr} & b_{22}^{sr} & b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{rr} & 0 & y_1^{sr} & 0 \\ 0 & y_2^{rr} & 0 & y_2^{rs} \\ y_1^{rs} & 0 & y_1^{ss} & 0 \\ 0 & y_2^{rs} & 0 & y_2^{ss} \end{bmatrix} \tag{8}$$

which we use to obtain exposure for all country/sectors. The framework can be directly generalized for G countries and N sectors.